

Homework 2

1.2

1. a) $(2, +5) \cdot (5, 2) = -10 + 10 = \boxed{0}$ so $\theta = \frac{\pi}{2}$
- b) $(2, 1) \cdot (-1, 1) = -2 + 1 = \boxed{-1}$ then $\theta = \cos^{-1}\left(\frac{-1}{\sqrt{10}}\right)$
- c) $(1, 4, -3) \cdot (5, 1, 3) = 5 + 4 - 9 = \boxed{0}$ so $\theta = \frac{\pi}{2}$
- d) $(1, -1, 6) \cdot (5, 3, 2) = 5 - 3 + 12 = \boxed{14}$ $\theta = \cos^{-1}\left(\frac{14}{\sqrt{38}}\right)$
- e) $(1, 1, 1, 1) \cdot (1, -3, -1, 5) = \boxed{2}$ then $\theta = \cos^{-1}\left(\frac{1}{\sqrt{6}}\right)$

2. a) $\text{proj}_{\bar{y}} \bar{x} = \bar{0}$; $\text{proj}_{\bar{x}} \bar{y} = \bar{0}$

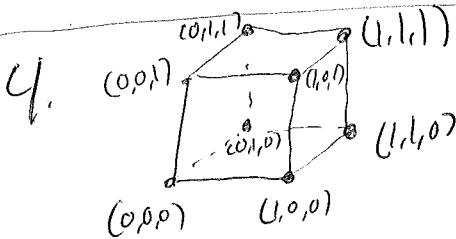
b) $\text{proj}_{\bar{y}} \bar{x} = \frac{(2, 1, 1) \cdot (-1, 1)}{1+1} \bar{y} = -\frac{1}{2} \bar{y}$
 $\text{proj}_{\bar{x}} \bar{y} = \frac{(2, 1) \cdot (-1, 1)}{1+4} \bar{x} = -\frac{1}{5} \bar{x}$

c) $\text{proj}_{\bar{y}} \bar{x} = \bar{0}$; $\text{proj}_{\bar{x}} \bar{y} = \bar{0}$

$$\text{proj}_{\bar{x}} \bar{y} = \frac{14}{1+1+36} \bar{x} = \boxed{\frac{7}{19} \bar{x}}$$

d) $\text{proj}_{\bar{y}} \bar{x} = \frac{14}{25+9+4} \bar{y} = \boxed{\frac{14}{38} \bar{y}}$

e) $\text{proj}_{\bar{y}} \bar{x} = \frac{2}{1+9+1+25} \bar{y} = \frac{1}{18} \bar{y}$ $\text{proj}_{\bar{x}} \bar{y} = \frac{2}{1+1+1} \bar{x} = \boxed{\frac{1}{3} \bar{x}}$



so angle between $(1,1,1)$ and $(1,1,0)$:

$$\theta = \cos^{-1}\left(\frac{(1,1,1) \cdot (1,1,0)}{\|(1,1,1)\| \cdot \|(1,1,0)\|}\right)$$

$$\theta = \cos^{-1}\left(\frac{2}{\sqrt{3}\sqrt{2}}\right) = \boxed{\cos^{-1}\left(\frac{2}{\sqrt{6}}\right)}$$

7. We know: $\bar{x} \cdot \bar{y} = \|\bar{x}\| \cdot \|\bar{y}\| \cos \theta$
 $= (\sqrt{2})(1) \cos(\frac{3\pi}{4}) = -1$

Then $(2\bar{x} + 3\bar{y}) \cdot (\bar{x} - \bar{y}) = 2\bar{x} \cdot \bar{x} - 2\bar{x} \cdot \bar{y} + 3\bar{x} \cdot \bar{y} - 3\bar{y} \cdot \bar{y}$
 $= 2\|\bar{x}\|^2 + \bar{x} \cdot \bar{y} - 3\|\bar{y}\|^2$
 $= 2(2) + (-1) - 3 = 0.$

So $2\bar{x} + 3\bar{y}$ and $\bar{x} - \bar{y}$ are orthogonal.

10. $\|\bar{x}\|^2 = n$ and $\|\bar{y}\|^2 = \frac{n(n+1)(2n+1)}{6}$.

Also $\bar{x} \cdot \bar{y} = \frac{n(n+1)}{2}$

So $\cos \theta_n = \frac{\frac{n(n+1)}{2}}{\sqrt{\frac{n(n+1)(2n+1)}{6}}} = \sqrt{\frac{3(n+1)}{2(2n+1)}}$

as $n \rightarrow \infty$ $\cos \theta_n \rightarrow \frac{\sqrt{3}}{\sqrt{4}} = \frac{\sqrt{3}}{2}$ So $\boxed{\theta_n \rightarrow \frac{\pi}{6} \text{ as } n \rightarrow \infty}$

11. Let $\bar{x}, \bar{v}_1, \dots, \bar{v}_k \in \mathbb{R}^n$ and suppose $\bar{x} \cdot \bar{v}_i = 0$ for all i .

Let $\bar{y} = c_1 \bar{v}_1 + \dots + c_k \bar{v}_k$ be an arbitrary linear combination of $\bar{v}_1, \dots, \bar{v}_k$.

$$\begin{aligned} \bar{x} \cdot \bar{y} &= \bar{x} \cdot (c_1 \bar{v}_1 + \dots + c_k \bar{v}_k) \\ \text{Then } \bar{x} \cdot \bar{y} &= \bar{x} \cdot (c_1 \bar{v}_1) + \dots + \bar{x} \cdot (c_k \bar{v}_k) \\ &= c_1 (\bar{x} \cdot \bar{v}_1) + \dots + c_k (\bar{x} \cdot \bar{v}_k) \\ &= c_1 (0) + \dots + c_k (0) = 0 \end{aligned}$$

So \bar{x} is orthogonal to \bar{y} . Since \bar{y} was arbitrary, \bar{x} is orthogonal to any linear combination of $\bar{v}_1, \dots, \bar{v}_k$.

16. a) Let $\bar{y} \in \mathbb{R}^n$. Suppose $\bar{x} \cdot \bar{y} = 0$ for all $\bar{x} \in \mathbb{R}^n$.

3.

In particular, if we let $\bar{x} = \bar{y}$, then we know $\bar{y} \cdot \bar{y} = 0$.

This means $\bar{y} = \bar{0}$ as required.

b) Suppose $\bar{y}, \bar{z} \in \mathbb{R}^n$ at $\bar{y} \cdot \bar{x} = \bar{z} \cdot \bar{x}$ for all $\bar{x} \in \mathbb{R}^n$.

Let $\bar{\omega} = \bar{y} - \bar{z}$. Then $\bar{\omega} \cdot \bar{x} = (\bar{y} - \bar{z}) \cdot \bar{x} = \bar{y} \cdot \bar{x} - \bar{z} \cdot \bar{x} = 0$
for all $\bar{x} \in \mathbb{R}^n$. By a) $\bar{\omega} = \bar{0}$. Thus $\bar{y} - \bar{z} = \bar{0}$ so

$$\bar{y} = \bar{z}.$$

1.3

a) $\bar{a} = (-2, 3)$ so $\bar{a} \cdot \bar{x} = \bar{a} \cdot (-1, 2)$

$$\rightarrow (-2, 3) \cdot \bar{x} = (-2, 3) \cdot (-1, 2)$$

$$\rightarrow \boxed{-2x_1 + 3x_2 = 8}$$

b) We know $(-1, 1, -1)$ is normal to the plane at $(1, 2, 2)$ is on plane so

$$\underline{(-1, 1, -1) \cdot \bar{x} = (-1, 1, -1) \cdot (1, 2, 2)}$$

$$\rightarrow \boxed{-x_1 + x_2 - x_3 = -1}$$

c) We know $(1, 2, -2)$ is normal to plane and $(2, 0, 1)$ is on plane

$$\underline{(1, 2, -2) \cdot \bar{x} = (1, 2, -2) \cdot (2, 0, 1)}$$

$$\rightarrow \boxed{x_1 + 2x_2 - 2x_3 = 0}$$

d) \bar{a} orthogonal to $(1, 1, 1)$ and $(2, 1, 0)$ is given by

$$a_1 + a_2 + a_3 = 0 \text{ and } 2a_1 + a_2 = 0$$

$$\text{so } a_3 = a_1 + a_2 \text{ and } a_1 = -\frac{1}{2}a_2$$

$$\text{in particular we can use } \bar{a} = (-1, 2, -1).$$

$$\text{so } \underline{(-1, 2, -1) \cdot \bar{x} = (-1, 2, -1) \cdot (1, 1, 2)}$$

$$\rightarrow \boxed{-x_1 + 2x_2 - x_3 = -1}$$

e) \bar{a} normal to $(1, 0, 1)$ and $(1, 2, 2)$ satisfies $a_1 + a_3 = 0$ and $a_1 + 2a_2 + a_3 = 0$

$$\text{so take } \bar{a} = (2, 1, -2).$$

$$\text{The } \underline{(2, 1, -2) \cdot \bar{x} = (2, 1, -2) \cdot (-1, 1, 1)}$$

$$\rightarrow \boxed{2x_1 + x_2 - 2x_3 = -3}$$

1. (continued)

f) \bar{a} normal to $(1, -1, 1, -1)$ at $(1, 1, -1, -1)$ at $(1, -1, 1, 1)$

$$\text{satisfies } \begin{aligned} a_1 - a_2 + a_3 - a_4 &= 0 & 2a_1 - 2a_4 &= 0 \\ a_1 + a_2 - a_3 - a_4 &= 0 & 2a_1 - 2a_2 &= 0 \\ a_1 - a_2 - a_3 + a_4 &= 0 & \text{and so } a_1 = a_4 \quad \text{and } a_1 = a_3 \\ && a_1 = a_2 \end{aligned}$$

pick $\bar{a} = (1, 1, 1, 1)$

$$\text{then } \bar{a} \cdot \bar{x} = 0 \text{ so } [x_1 + x_2 + x_3 + x_4 = 0]$$

$$3. \text{ a) } \bar{x} = (4+2x_2-3x_3, x_2, x_3)$$

$$= [(4, 0, 0) + x_2 (2, 1, 0) + x_3 (-3, 0, 1)]$$

$$\text{c) } \bar{x} = (5-x_2+x_3-2x_4, x_2, x_3, x_4)$$

$$= [(5, 0, 0, 0) + x_2 (-1, 1, 0, 0) + x_3 (1, 0, 1, 0) + x_4 (-2, 0, 0, 1)]$$

$$\text{e) } \bar{x} = (x_1, 2-x_3+3x_4, x_3, x_4)$$

$$= [(0, 2, 0, 0) + x_1 (1, 0, 0, 0) + x_3 (0, -1, 1, 0) + x_4 (0, 3, 0, 1)]$$

9. a) pick $\bar{a} = (2, 2, -3, 8)$ b/c it is given.

$$\text{b) } \text{proj}_{\bar{a}} \bar{x} = \frac{\bar{x} \cdot \bar{a}}{\|\bar{a}\|^2} \bar{a} = \frac{6}{81} \bar{a} \quad \text{also } \left| \frac{6}{81} \bar{a} \right| = \frac{6}{81} \cdot 9 = \frac{6}{9} = \boxed{\frac{2}{3}}$$

c) line through origin with direction \bar{a} is $\bar{x} = (2, 2, -3, 8)t$ This intersects hyperplane when $\bar{a} \cdot (2, 2, -3, 8)t = 6$ i.e. when $8t = 6$ so $t = \frac{6}{81}$ or $\frac{2}{27}$. This closest point on plane is $\frac{2}{27}(2, 2, -3, 8)$. Then $\left\| \frac{2}{27}(2, 2, -3, 8) \right\| = \frac{2}{3}$ as b) said.

q. continued

d) Project $(1,1,1,1)$ onto \bar{a} + set $\text{proj}_{\bar{a}}(1,1,1,1) = \frac{(2,2,-3,8)(1,1,1,1)}{\|a\|^2} \bar{a}$

$$= \frac{9}{81} \bar{a}$$

at place projects + $\frac{6}{81} \bar{a}$ so distance between them is

$$\left\| \frac{9}{81} \bar{a} - \frac{6}{81} \bar{a} \right\| = \left\| \frac{3}{81} \bar{a} \right\| = \frac{3 \cdot 9}{81} = \frac{3}{9} = \boxed{\frac{1}{3}}$$

e) Line through $(1,1,1,1)$ with direction $\bar{a} = (2,2,-3,8)$ is

$$\bar{x} = (1,1,1,1) + t(2,2,-3,8).$$

This intersects hyperplane when

$$\bar{a} \cdot ((1,1,1,1) + t(2,2,-3,8)) = 6$$

$$\bar{a} \cdot ((1,1,1,1) + t(2,2,-3,8)) = 6$$

$$\text{so } 9 + t \cdot 81 = 6 \rightarrow t = -\frac{3}{81} = -\frac{1}{27}$$

$$\text{so point of intersection is } (1,1,1,1) + \frac{1}{27}(2,2,-3,8) = \bar{x}$$

So point of intersection is $(1,1,1,1) + \frac{1}{27}(2,2,-3,8)$.

distance from this point to $(1,1,1,1)$ is

$$\left\| (1,1,1,1) - \left[(1,1,1,1) + \frac{1}{27}(2,2,-3,8) \right] \right\| = \left\| \frac{1}{27}(2,2,-3,8) \right\|$$

$$= \frac{1}{27} \cdot 9 = \boxed{\frac{1}{3}}$$

as in d).

(2. a) Suppose $\bar{a} \neq 0$ at \mathcal{P} is plane with normal vector \bar{a}
and spanned by \bar{u} and \bar{v} . 7.

Suppose $\bar{u} \cdot \bar{v} = 0$. Let $\bar{x} \in \mathcal{P}$. Then $\bar{x} = s\bar{u} + t\bar{v}$ where

$s, t \in \mathbb{R}$.

$$\text{Now } \bar{x} \cdot \bar{u} = (\bar{s}\bar{u} + t\bar{v}) \cdot \bar{u} = s\bar{u} \cdot \bar{u} + t\bar{v} \cdot \bar{u} = s\|\bar{u}\|^2 + 0$$

$$\text{So } s = \frac{\bar{x} \cdot \bar{u}}{\|\bar{u}\|^2}$$

$$\text{Also } \bar{x} \cdot \bar{v} = (\bar{s}\bar{u} + t\bar{v}) \cdot \bar{v} = s\bar{u} \cdot \bar{v} + t\bar{v} \cdot \bar{v} = 0 + t\|\bar{v}\|^2$$

$$\text{So } t = \frac{\bar{x} \cdot \bar{v}}{\|\bar{v}\|^2}$$

Thus $\bar{x} = \left(\frac{\bar{x} \cdot \bar{u}}{\|\bar{u}\|^2}\right)\bar{u} + \left(\frac{\bar{x} \cdot \bar{v}}{\|\bar{v}\|^2}\right)\bar{v}$ and by definition this

means $\bar{x} = \text{proj}_{\bar{u}} \bar{x} + \text{proj}_{\bar{v}} \bar{x}$.

b). Suppose $\bar{u} \cdot \bar{v} = 0$. we also know $\bar{a} \cdot \bar{u} = 0$ and $\bar{a} \cdot \bar{v} = 0$ by
assumption above.

Let $\bar{x} \in \mathbb{R}^3$. Define $\bar{x} - \text{proj}_{\bar{a}} \bar{x} := \bar{\omega}$.

Claim: $\bar{\omega} \in \mathcal{P}$.

$$\begin{aligned} \text{Proof: } \bar{\omega} \cdot \bar{a} &= (\bar{x} - \text{proj}_{\bar{a}} \bar{x}) \cdot \bar{a} \\ &= \bar{x} \cdot \bar{a} - \frac{\bar{x} \cdot \bar{a}}{\|\bar{a}\|^2} \cdot \bar{a} \cdot \bar{a} = \bar{x} \cdot \bar{a} - \frac{\bar{x} \cdot \bar{a}}{\|\bar{a}\|^2} \|\bar{a}\|^2 \\ &= 0. \end{aligned}$$

So $\bar{\omega} \in \mathcal{P}$.

Then, by a) $\bar{\omega} = \text{proj}_{\bar{u}} \bar{\omega} + \text{proj}_{\bar{v}} \bar{\omega}$.

next →

12 b) continued...

8.

Since $\bar{\omega} = \bar{x} - \text{proj}_{\bar{a}} \bar{x}$, we have

$$\bar{x} - \text{proj}_{\bar{a}} \bar{x} = \text{proj}_{\bar{u}} (\bar{x} - \text{proj}_{\bar{a}} \bar{x}) + \text{proj}_{\bar{v}} (\bar{x} - \text{proj}_{\bar{a}} \bar{x})$$

$$\text{So } \bar{x} = \text{proj}_{\bar{a}} \bar{x} + \text{proj}_{\bar{u}} \left(\bar{x} - \left(\frac{\bar{x} \cdot \bar{a}}{\|\bar{a}\|^2} \right) \bar{a} \right) + \text{proj}_{\bar{v}} \left(\bar{x} - \left(\frac{\bar{x} \cdot \bar{a}}{\|\bar{a}\|^2} \right) \bar{a} \right)$$

$$= \text{proj}_{\bar{a}} \bar{x} + \underbrace{\left(\bar{x} - \left(\frac{\bar{x} \cdot \bar{a}}{\|\bar{a}\|^2} \right) \bar{a} \right) \cdot \bar{u}}_{\|\bar{u}\|^2} \bar{u} + \underbrace{\frac{\left(\bar{x} - \left(\frac{\bar{x} \cdot \bar{a}}{\|\bar{a}\|^2} \right) \bar{a} \right) \cdot \bar{v}}{\|\bar{v}\|^2} \bar{v}}$$

$$= \text{proj}_{\bar{a}} \bar{x} + \left(\frac{\bar{x} \cdot \bar{u}}{\|\bar{u}\|^2} + \frac{\left(\bar{x} \cdot \bar{a} \right) \bar{a} \cdot \bar{u}}{\|\bar{u}\|^2} \right) \bar{u} + \left(\frac{\bar{x} \cdot \bar{v} - \left(\frac{\bar{x} \cdot \bar{a}}{\|\bar{a}\|^2} \right) \bar{a} \cdot \bar{v}}{\|\bar{v}\|^2} \right) \bar{v}$$

and since $\bar{a} \cdot \bar{u} = 0$ and $\bar{a} \cdot \bar{v} = 0$, we get

$$\bar{x} = \text{proj}_{\bar{a}} \bar{x} + \frac{\bar{x} \cdot \bar{u}}{\|\bar{u}\|^2} \bar{u} + \frac{\bar{x} \cdot \bar{v}}{\|\bar{v}\|^2} \bar{v}$$

$$= \text{proj}_{\bar{a}} \bar{x} + \text{proj}_{\bar{u}} \bar{x} + \text{proj}_{\bar{v}} \bar{x}$$

as required.